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# INFORMATION CONTENT OF PARTIALLY RANK-ORDERED SET SAMPLES

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## Abstract:

Partially rank-ordered set (PROS) sampling is a generalization of ranked set sampling in which rankers are not required to fully rank the sampling units in each set, hence having more flexibility to perform the necessary judgemental ranking process. The PROS sampling has a wide range of applications in different fields ranging from environmental and ecological studies to medical research and it has been shown to be superior over ranked set sampling and simple random sampling for estimating the population mean. We study the Fisher information content and uncertainty structure of the PROS samples and compare them with those of simple random sample (SRS) and ranked set sample (RSS) counterparts of the same size from the underlying population. We study the uncertainty structure in terms of the Shannon entropy, Rényi entropy and Kullback-Leibler (KL) discrimination measures. Several examples including the FI of PROS samples from the location-scale family of distributions as well as a regression model are discussed.

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## 1 Introduction

Ranked set sampling is a powerful and cost-effective data collection technique which can be used to obtain more representative samples from the underlying population when a small number of sampling units can be fairly accurately ordered with respect to a variable of interest without actual measurements on them and at little cost. It is assumed that the exact measurement of the variable of interest is very costly but ranking sampling units is cheap. Ranked set sampling has many applications in industrial statistics, environmental and ecological studies as well as medical research. Some recent examples include estimating phytomass (Muttalak and McDonald, 1992), stream habitat area (Mode et al., 1999), mean and variance in flock management (Ozturk et al., 2005) as well as studying the association between smoking exposure and three important carcinogenic biomarkers in a lung cancer decease study (Chen and Wang, 2004) and in a fishery research for estimating the mean stock abundance using the catch-rate data available from previous

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years as a concomitant variable (Wang et al., 2009). For recent overviews of the theory and applications of ranked set sampling and some of its variations see Wolfe (2012) and Chen et al. (2004).

To obtain a ranked set sample (RSS), an initial simple random sample (SRS) of size  $k$  is taken. These units are ordered, but without actually being measured; we call this judgement ranking, which may be perfect or imperfect. Upon ranking, only the smallest unit is measured. Following this, a second SRS of size  $k$  is taken, ranked and the second smallest unit is measured. This process is repeated until the largest unit in a SRS of size  $k$  has been measured. In this process, the ranker is asked to declare unique ranks for each unit inside the sets. There are many situations where it is difficult to rank all of the sampling units in a set with high confidence, particularly when subjective information is utilized in the ranking process. Forcing rankers to declare unique ranks can lead to inflated within-set judgment ranking error and consequently to invalid statistical inference. Partially rank-ordered set (PROS) sampling design is a generalization of ranked set sampling, due to Ozturk (2011), which is aimed at reducing the impact of ranking error and the burden on rankers by not requiring them to provide a full ranking of all the units in each set. Under PROS sampling technique, rankers have more flexibility by being able to divide the sampling units into subsets of pre-specified sizes. These subsets are partially rank-ordered so that each unit in subset  $h$  has a rank smaller than the ranks of units in subset  $h'$  for all  $h' \geq h$ . An observation is then collected from one of these subsets in each set. Hatefi et al. (2015) used PROS sampling design to estimate the parameters of a finite mixture model to analyze the age structure of a fish species. Frey (2012) studied nonparametric mean estimation using PROS sampling design. Ozturk (2013) proposed statistical procedures that utilize PROS data from multiple observers to assist in the selection of units for measurement in a basic ranked set sample design or to construct a judgment post-stratified design.

In this paper, we study information and uncertainty content of PROS samples. To this end, in Section 2, we provide a formal description of PROS sampling and present some preliminary results on distributional properties of PROS samples. In Section 3, we obtain the Fisher information (FI) content of PROS samples and show that it is more than the FI content of its SRS and RSS counterparts of the same size. Several examples including the FI of PROS samples from the location-scale family of distributions as well as a simple linear regression model are also discussed in this section. In addition, the effect of subsetting errors when applying PROS sampling design on the FI content of samples is explored. In Section 4, we study information and uncertainty of PROS samples using the Shannon entropy, Rényi entropy and KL information measures and compare them with their SRS and RSS counterparts. Finally, in Section 5, we give some concluding remarks.

## 2 Preliminary results on distributional properties of PROS samples

To obtain a PROS sample of size  $n$ , we choose a set size  $S$  and a design parameter  $D = \{d_1, \dots, d_n\}$  that partitions the set  $\{1, \dots, S\}$  into  $n$  mutually exclusive subsets. First,  $S$  units are randomly selected and are assigned into subsets  $d_r, r = 1, \dots, n$ , without actual measurement of the variable of interest and only based on visual inspection or judgment, etc. Then a unit is selected at random for measurement from the subset  $d_1$  and it is denoted by  $X_{(d_1)1}$ . Selecting another  $S$  units assigning them into subsets, a unit is randomly drawn from subset  $d_2$  and then it is quantified and denoted by  $X_{(d_2)1}$ . This process is repeated until we randomly draw a unit from  $d_n$  resulting in  $X_{(d_n)1}$ . This constitutes one *cycle* of PROS sampling technique. The cycle is then repeated  $N$  times to generate a PROS sample of the size  $Nn$ , i.e.  $\{X_{(d_r)i}; r = 1, \dots, n; i = 1, \dots, N\}$ . Table 1 shows the construction of a balanced PROS sample with  $S = 6, n = 2, N = 2$  and the design parameter  $D = \{d_1, d_2\} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$ . Each set includes six units assigned into two partially ordered subsets. This partial ordering provides the information that the units in  $d_1$  have the smaller ranks than units in  $d_2$ . In this subsetting process we do not assign any ranks to units within each subset so that these units are equally likely to take any place in the subset. One unit, in each set from the bold faced subset, is randomly drawn and is quantified. The fully measured units are denoted by  $X_{(d_r)i}, r = 1, 2; i = 1, 2$ .

Table 1: An example of PROS design

cycle	set	Subsets	Observation
1	$S_1$	$D_1 = \{d_1, d_2\} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$	$X_{(d_1)1}$
	$S_2$	$D_2 = \{d_1, d_2\} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$	$X_{(d_2)1}$
2	$S_1$	$D_1 = \{d_1, d_2\} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$	$X_{(d_1)2}$
	$S_2$	$D_2 = \{d_1, d_2\} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$	$X_{(d_2)2}$

Throughout the paper, without loss of generality, we assume that  $N = 1$  (unless otherwise specified) and we use  $\text{PROS}(n, S)$  to denote a PROS sampling design with the set size  $S$ , the number of subsets  $n$  and the design parameter  $D = \{d_r, r = 1, \dots, n\}$  where  $d_r = \{(r-1)m + 1, \dots, rm\}$ , in which  $m = S/n$  is the number of unranked observations in each subset. We note that RSS and SRS can be expressed as special cases of the  $\text{PROS}(n, S)$  design when  $S = n$  and  $S = 1$ , respectively.

Suppose  $X$  is a continuous random variable with probability density function (pdf)  $f(x; \theta)$  and cumulative density function (cdf)  $F(x; \theta)$ , where  $\theta$  is the vector of unknown parameters with  $\theta \in \mathbb{R}^p$ . Let  $\mathbf{X}_{\text{pros}} = \{X_{(d_r)}, r = 1, \dots, n\}$  be a perfect  $\text{PROS}(n, S)$  sample of size  $n$  from  $f(\cdot, \theta)$ . The PROS data

likelihood function of  $\boldsymbol{\theta}$  is given by the joint pdf of  $\mathbf{X}_{pros}$  as follows:

$$L(\boldsymbol{\theta}|\mathbf{x}_{pros}) = f(\mathbf{x}_{pros}; \boldsymbol{\theta}) = \prod_{r=1}^n \left\{ \frac{1}{m} \sum_{u \in d_r} f^{(u:S)}(x_{(d_r)}; \boldsymbol{\theta}) \right\},$$

where  $f^{(u:S)}(\cdot; \boldsymbol{\theta})$  is the pdf of the  $u$ -th order statistic of a SRS of size  $S$  from  $f(\cdot; \boldsymbol{\theta})$ . For each  $X_{(d_r)}$  define the latent vector  $\boldsymbol{\Delta}^{(d_r)} = (\Delta^{(d_r)}(u), u \in d_r = \{(r-1)m+1, \dots, rm\})$ , where

$$\Delta^{(d_r)}(u) = \begin{cases} 1 & \text{if } X_{(d_r)} \text{ is selected from the } u\text{-th position within the subset } d_r; \\ 0 & \text{otherwise,} \end{cases}$$

with  $\sum_{u \in d_r} \Delta^{(d_r)}(u) = 1$ . Denote  $\mathbf{Y}_{pros} = \{(X_{(d_r)}, \boldsymbol{\Delta}^{(d_r)}), r = 1, \dots, n\}$  as the complete PROS data consisting of  $X_{(d_r)}$  and their corresponding latent vectors  $\boldsymbol{\Delta}^{(d_r)}$ ,  $r = 1, \dots, n$ . The complete PROS data likelihood function of  $\boldsymbol{\theta}$  using the joint pdf of  $\mathbf{Y}_{pros}$  is given by

$$L(\boldsymbol{\theta}|\mathbf{y}_{pros}) = f(\mathbf{y}_{pros}; \boldsymbol{\theta}) = \prod_{r=1}^n \prod_{u \in d_r} \left\{ \frac{1}{m} f^{(u:S)}(x_{(d_r)}; \boldsymbol{\theta}) \right\}^{\boldsymbol{\delta}^{(d_r)}(u)}. \quad (1)$$

Furthermore, by summing the joint distribution of  $(X_{(d_r)}, \boldsymbol{\Delta}^{(d_r)})$  over  $\boldsymbol{\Delta}^{(d_r)} = \boldsymbol{\delta}^{(d_r)}$ , the marginal distribution of  $X_{(d_r)}$  is obtained as follows

$$f_{(d_r)}(x_{(d_r)}; \boldsymbol{\theta}) = \sum_{\boldsymbol{\delta}^{(d_r)}} f(x_{(d_r)}, \boldsymbol{\delta}^{(d_r)}; \boldsymbol{\theta}) = \frac{1}{m} \sum_{u \in d_r} f^{(u:S)}(x_{(d_r)}; \boldsymbol{\theta}). \quad (2)$$

Also, one can easily check that

$$\frac{1}{n} \sum_{r=1}^n f_{(d_r)}(x; \boldsymbol{\theta}) = f(x; \boldsymbol{\theta}). \quad (3)$$

In addition, the conditional distribution of  $\boldsymbol{\Delta}^{(d_r)}$  given  $X_{(d_r)}$  is

$$f(\boldsymbol{\delta}^{(d_r)}|x_{(d_r)}; \boldsymbol{\theta}) = \prod_{u \in d_h} \left\{ \frac{f^{(u:S)}(x_{(d_r)}; \boldsymbol{\theta})}{\sum_{v \in d_r} f^{(v:S)}(x_{(d_r)}; \boldsymbol{\theta})} \right\}^{\boldsymbol{\delta}^{(d_r)}(u)}. \quad (4)$$

### 3 FI content of PROS samples

In this section, we first obtain the FI content of  $\mathbf{Y}_{pros}$ , the complete PROS data, and derive analytic results to compare it with the FI content of SRS and RSS data of the same size. We give examples regarding the location-scale family of distributions as well as a simple linear regression model. Then, we study the FI content of  $\mathbf{X}_{pros}$  by modelling an imperfect PROS design involving misplacement errors in the subsetting process. The FI of PROS samples can play a key role in its theory and application to study the asymptotic

behaviour of the maximum likelihood estimators of  $\boldsymbol{\theta}$  as well as the derivation of the Cramer-Rao lower bound for unbiased estimators of  $\boldsymbol{\theta}$  or some of its functions based on PROS samples.

Under the usual regularity conditions (e.g., Chen et al., 2004), the FI matrix is calculated by  $\mathbb{I}(\boldsymbol{\theta}) = -\mathbb{E}[D_{\boldsymbol{\theta}}^2 \log f(X; \boldsymbol{\theta})]$ , provided the expectation exists, where  $D_{\boldsymbol{\theta}}^l$  refers to the  $l$ -th derivatives of the log-likelihood function with respect to  $\boldsymbol{\theta}$  with  $D_{\boldsymbol{\theta}}^1 = D_{\boldsymbol{\theta}}$ . For any two matrices  $A$  and  $B$  of the same size, we use  $A \geq 0$  and  $A \geq B$  to indicate that  $A$  and  $A - B$  are non-negative definite matrices. We also let  $\phi_u(\lambda) = (u - 1)I(\lambda = 0) + (S - u)I(\lambda = 1)$  with  $\lambda \in \{0, 1\}$ ,  $u = 1, \dots, S$ , where  $I$  is the usual indicator function.

### 3.1 FI matrix of complete PROS data $\mathbf{Y}_{pros}$

Here we obtain the FI matrix of  $\mathbf{Y}_{pros}$  under perfect subsetting assumption. To do so, we need the following useful result.

**Lemma 1.** *Suppose  $Y_r = X_{(d_r)}$ , with pdf  $f_{(d_r)}(\cdot; \boldsymbol{\theta})$ , is observed from a continuous distribution with pdf  $f(\cdot; \boldsymbol{\theta})$  and cdf  $F(\cdot; \boldsymbol{\theta})$ , respectively, using a PROS( $n, S$ ) design. Let  $\boldsymbol{\delta}^{(d_r)}(u)$  be the latent variable associated with  $X_{(d_r)}$ . For any  $\lambda \in \{0, 1\}$  and any function  $G(\cdot)$ ,*

$$\mathbb{E} \left\{ \sum_{r=1}^n \sum_{u \in d_r} \frac{\phi_u(\lambda) \boldsymbol{\delta}^{(d_r)}(u) G(Y_r)}{\lambda + (1 - 2\lambda) F(Y_r; \boldsymbol{\theta})} \right\} = n(S - 1) \mathbb{E}[G(X)],$$

*subject to the existence of the expectations.*

*Proof.* Let  $\lambda = 0$ . By the total law of expectations and equation (4) we get

$$\begin{aligned} \mathbb{E} \left\{ \sum_{r=1}^n \sum_{u \in d_r} (u - 1) \frac{\boldsymbol{\delta}^{(d_r)}(u) G(Y_r)}{F(Y_r; \boldsymbol{\theta})} \right\} &= \frac{1}{m} \sum_{r=1}^n \sum_{u \in d_r} (u - 1) \int \frac{G(x)}{F(x; \boldsymbol{\theta})} f^{(u:S)}(x; \boldsymbol{\theta}) dx \\ &= \frac{S}{m} \int G(x) f(x; \boldsymbol{\theta}) \left\{ \sum_{v=1}^S (v - 1) \binom{S - 1}{v - 1} [F(x; \boldsymbol{\theta})]^{v-2} [\bar{F}(x; \boldsymbol{\theta})]^{S-v} \right\} dx \\ &= n(S - 1) \mathbb{E}[G(X)], \end{aligned}$$

The proof for  $\lambda = 1$  is similar and hence is omitted. □

Now, we obtain the FI content of  $\mathbf{Y}_{pros}$  and compare it with its SRS counterpart of the same size.

**Theorem 1.** *Under the usual regularity conditions (e.g., Chen et al., 2004), the FI matrix of a complete PROS( $n, S$ ) sample of size  $n$  from  $f(\cdot; \boldsymbol{\theta})$  is given by*

$$\mathbb{I}_{pros}(\boldsymbol{\theta}) = \mathbb{I}_{srs}(\boldsymbol{\theta}) + \mathbb{K}(\boldsymbol{\theta}),$$

where  $\mathbb{I}_{srs}(\boldsymbol{\theta})$  denotes the FI matrix of a SRS of size  $n$ ,

$$\mathbb{K}(\boldsymbol{\theta}) = n(S-1)\mathbb{E} \left\{ \frac{[D_{\boldsymbol{\theta}}F(X; \boldsymbol{\theta})][D_{\boldsymbol{\theta}}F(X; \boldsymbol{\theta})]^\top}{F(X; \boldsymbol{\theta})\bar{F}(X; \boldsymbol{\theta})} \right\},$$

is a non-negative definite matrix and the expectation is taken with respect to  $X$ .

*Proof.* Let  $Y_r = X_{(d_r)}$ ,  $r = 1, \dots, n$ . Using (1), the log-likelihood function of  $\boldsymbol{\theta}$  can be written as

$$l_{pros}(\boldsymbol{\theta}) = cst + l_{srs}^*(\boldsymbol{\theta}) + \Gamma_p(\boldsymbol{\theta}),$$

where  $cst = n \log\{n \binom{S-1}{n-1}\}$  is a constant with respect to  $\boldsymbol{\theta}$  and

$$\Gamma_p(\boldsymbol{\theta}) = \sum_{r=1}^n \sum_{u \in d_r} \sum_{\lambda=0}^1 \phi_u(\lambda) \delta^{(d_r)}(u) \log[\lambda + (1-2\lambda)F(y_r; \boldsymbol{\theta})],$$

and  $-\mathbb{E}[D_{\boldsymbol{\theta}}^2 l_{srs}^*(\boldsymbol{\theta})] = \mathbb{I}_{srs}(\boldsymbol{\theta})$ . Taking second derivatives of  $\Gamma_p(\boldsymbol{\theta})$  with respect to  $\boldsymbol{\theta}$ , one gets

$$D_{\boldsymbol{\theta}}^2 \Gamma_p(\boldsymbol{\theta}) = \sum_{r=1}^n \sum_{u \in d_r} \sum_{\lambda=0}^1 \phi_u(\lambda) \delta^{(d_r)}(u) \left\{ \frac{(-1)^\lambda D_{\boldsymbol{\theta}}^2 F(y_r; \boldsymbol{\theta})}{\lambda + (1-2\lambda)F(y_r; \boldsymbol{\theta})} - \frac{[D_{\boldsymbol{\theta}}F(y_r; \boldsymbol{\theta})][D_{\boldsymbol{\theta}}F(y_r; \boldsymbol{\theta})]^\top}{[\lambda + (1-2\lambda)F(y_r; \boldsymbol{\theta})]^2} \right\}.$$

Using Lemma 1, we have

$$\begin{aligned} \mathbb{E} \left\{ \sum_{r=1}^n \sum_{u \in d_r} (u-1) \frac{\delta^{(d_r)}(u) D_{\boldsymbol{\theta}}^2 F(Y_r; \boldsymbol{\theta})}{F(Y_r; \boldsymbol{\theta})} \right\} &= \mathbb{E} \left\{ \sum_{r=1}^n \sum_{u \in d_r} (S-u) \frac{\delta^{(d_r)}(u) D_{\boldsymbol{\theta}}^2 F(Y_r; \boldsymbol{\theta})}{\bar{F}(Y_r; \boldsymbol{\theta})} \right\} \\ &= S(S-1)\mathbb{E} \{ D_{\boldsymbol{\theta}}^2 F(X; \boldsymbol{\theta}) \}. \end{aligned} \quad (5)$$

Similarly, by Lemma 1, we obtain

$$\begin{aligned} \mathbb{E} \left\{ \sum_{r=1}^n \sum_{u \in d_r} \phi_u(\lambda) \delta^{(d_r)}(u) \times \frac{[D_{\boldsymbol{\theta}}F(Y_r; \boldsymbol{\theta})][D_{\boldsymbol{\theta}}F(Y_r; \boldsymbol{\theta})]^\top}{[\lambda + (1-2\lambda)F(Y_r; \boldsymbol{\theta})]^2} \right\} \\ = n(S-1)\mathbb{E} \left\{ \frac{[D_{\boldsymbol{\theta}}F(X; \boldsymbol{\theta})][D_{\boldsymbol{\theta}}F(X; \boldsymbol{\theta})]^\top}{\lambda + (1-2\lambda)F(X; \boldsymbol{\theta})} \right\}, \quad \lambda \in \{0, 1\}. \end{aligned} \quad (6)$$

Taking expectation of  $D_{\boldsymbol{\theta}}^2 \Gamma_p(\boldsymbol{\theta})$  and from (5) and (6), we obtain

$$\mathbb{K}(\boldsymbol{\theta}) = -\mathbb{E}[D_{\boldsymbol{\theta}}^2 \Gamma_p(\boldsymbol{\theta})] = n(S-1)\mathbb{E} \left\{ \frac{[D_{\boldsymbol{\theta}}F(X; \boldsymbol{\theta})][D_{\boldsymbol{\theta}}F(X; \boldsymbol{\theta})]^\top}{F(X; \boldsymbol{\theta})\bar{F}(X; \boldsymbol{\theta})} \right\}, \quad (7)$$

which completes the proof.  $\square$

Theorem 1 shows that the FI matrix of the complete PROS( $n, S$ ) sample can be decomposed into the FI matrix of the SRS data and a non-negative definite matrix, hence  $\mathbb{I}_{pros}(\boldsymbol{\theta}) \geq \mathbb{I}_{srs}(\boldsymbol{\theta})$ . In other words, complete PROS sample provides more information about the unknown parameters  $\boldsymbol{\theta}$  than SRS of the same size. It is worth noting that the result of Chen (2000) and Barabesi and El-Sharaawi (2001) about FI of RSS data can be obtained a special case of Theorem 1 by setting  $S = n$ . We now compare the FI content of the complete PROS sample with that of RSS of the same size about the unknown parameters  $\boldsymbol{\theta}$ .

**Theorem 2.** *Under the conditions of Theorem 1, the FI matrix of a complete PROS( $n, S$ ) sample may be decomposed as*

$$\mathbb{I}_{pros}(\boldsymbol{\theta}) = \mathbb{I}_{rss}(\boldsymbol{\theta}) + \mathbb{H}(\boldsymbol{\theta}),$$

where  $\mathbb{I}_{rss}(\boldsymbol{\theta})$  is the FI matrix of an RSS of size  $n$  (when the set size is  $n$ ), and

$$\mathbb{H}(\boldsymbol{\theta}) = n(S - n)\mathbb{E} \left\{ \frac{[D_{\boldsymbol{\theta}}F(X; \boldsymbol{\theta})][D_{\boldsymbol{\theta}}F(X; \boldsymbol{\theta})]^\top}{F(X; \boldsymbol{\theta})\bar{F}(X; \boldsymbol{\theta})} \right\},$$

is a non-negative definite matrix.

*Proof.* Using Theorem 1 for  $S = n$ , we have

$$\mathbb{I}_{rss}(\boldsymbol{\theta}) = \mathbb{I}_{srs}(\boldsymbol{\theta}) + n(n - 1)\mathbb{E} \left\{ \frac{[D_{\boldsymbol{\theta}}F(X; \boldsymbol{\theta})][D_{\boldsymbol{\theta}}F(X; \boldsymbol{\theta})]^\top}{F(X; \boldsymbol{\theta})\bar{F}(X; \boldsymbol{\theta})} \right\},$$

where  $\mathbb{I}_{srs}(\boldsymbol{\theta})$  denotes the FI matrix of a SRS of size  $n$ . Now, the result follows from the above equation and the expression for  $\mathbb{I}_{pros}(\boldsymbol{\theta})$  in Theorem 1.  $\square$

Theorem 2 shows the superiority of a complete PROS sample over an RSS of the same size in terms of the FI content about the unknown vector of parameters  $\boldsymbol{\theta}$ . In comparing the Fisher information content of RSS data to that of SRS data, Barabesi and El-Sharaawi (2001) considered the example of point estimation within a location-scale family and the example of linear regression. We use the same two examples to obtain the FI content of a complete PROS data set from the location-scale family of distributions as well as a simple linear regression model and compare them with those based on SRS and RSS data of the same size. To this end, let

$$RE_1(\boldsymbol{\theta}) = \frac{\det\{\mathbb{I}_{pros}(\boldsymbol{\theta})\}}{\det\{\mathbb{I}_{srs}(\boldsymbol{\theta})\}} \quad \text{and} \quad RE_2(\boldsymbol{\theta}) = \frac{\det\{\mathbb{I}_{pros}(\boldsymbol{\theta})\}}{\det\{\mathbb{I}_{rss}(\boldsymbol{\theta})\}}.$$

From Theorems 1 and 2 one can notice that the set size ( $S$ ) and the number of the subsets ( $n$ ) are two important parameters of PROS( $n, S$ ) design that influence the FI content of PROS samples. We observe that increasing  $S$  and  $n$  results in a considerable gain in  $RE_1$  and  $RE_2$ , respectively. Also, both  $RE_1$  and  $RE_2$  increase with the number of the parameters of the model. Later in this section we investigate the case where the set sizes are fixed in both PROS and RSS designs and consider the effect of the number of subsets in PROS sampling design on the FI content of PROS data compared with their RSS counterparts.

**Example 1. (Location-Scale family of distributions).** *Under the assumptions of Theorem 1, if  $f(x; \boldsymbol{\theta})$  is a member of the location-scale family of distributions with pdf*

$$f(x; \boldsymbol{\theta}) = \frac{1}{\sigma}g\left(\frac{x - \mu}{\sigma}\right), \quad \boldsymbol{\theta} = (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^+,$$

where  $g(\cdot)$  is a pdf with corresponding cdf  $G(\cdot)$ , then

$$\mathbb{I}_{pros}(\boldsymbol{\theta}) = \mathbb{I}_{srs}(\boldsymbol{\theta}) + \mathbb{K}(\boldsymbol{\theta})$$

$$= \frac{n}{\sigma^2} \begin{pmatrix} E\left\{\frac{g'(Z)^2}{g(Z)^2}\right\} & E\left\{\frac{Zg'(Z)^2}{g(Z)^2}\right\} \\ E\left\{\frac{Zg'(Z)^2}{g(Z)^2}\right\} & E\left\{\frac{Z^2g'(Z)^2}{g(Z)^2} - 1\right\} \end{pmatrix} + \frac{n(S-1)}{\sigma^2} \begin{pmatrix} E\left\{\frac{g(Z)^2}{G(Z)[1-G(Z)]}\right\} & E\left\{\frac{Zg(Z)^2}{G(Z)[1-G(Z)]}\right\} \\ E\left\{\frac{Zg(Z)^2}{G(Z)[1-G(Z)]}\right\} & E\left\{\frac{Z^2g(Z)^2}{G(Z)[1-G(Z)]}\right\} \end{pmatrix}.$$

If  $f(x; \boldsymbol{\theta})$  is symmetric about the location parameter  $\mu$ , the FI matrix reduces to

$$\mathbb{I}_{pros}(\boldsymbol{\theta}) = \frac{n}{\sigma^2} \begin{pmatrix} E\left\{\frac{g'(Z)^2}{g(Z)^2}\right\} & 0 \\ 0 & E\left\{\frac{Z^2g'(Z)^2}{g(Z)^2} - 1\right\} \end{pmatrix} + \frac{n(S-1)}{\sigma^2} \begin{pmatrix} E\left\{\frac{g(Z)^2}{G(Z)[1-G(Z)]}\right\} & 0 \\ 0 & E\left\{\frac{Z^2g(Z)^2}{G(Z)[1-G(Z)]}\right\} \end{pmatrix}.$$

Similar to Barabesi and El-Sharaawi (2001) who compared the relative efficiency of RSS to SRS for some members of the location-scale family of distributions, Tables 2 shows the values of  $RE_1$  and  $RE_2$  under the same distributions. As expected, the largest values of  $RE_1$  and  $RE_2$  are achieved in the cases where both location and scale parameters are considered to be unknown.

Table 2: The values of  $RE_i(\boldsymbol{\theta})$ ,  $i = 1, 2$  for comparing the FI content of the complete PROS( $n, S$ ) sample with its SRS and RSS of the same size for some distributions.

Distributions	Location	Scale	Shape	$RE_1$	$RE_2$
Exponential	0	$\sigma$	-	$1+0.4041(S-1)$	$1+0.4041\left\{\frac{(S-n)}{1+0.4041(n-1)}\right\}$
Normal	$\mu$	1	-	$1+0.4805(S-1)$	$1+0.4805\left\{\frac{(S-n)}{1+0.4805(n-1)}\right\}$
	0	$\sigma$	-	$1+0.1350(S-1)$	$1+0.1350\left\{\frac{(S-n)}{1+0.1350(n-1)}\right\}$
	$\mu$	$\sigma$	-	$1+0.6155(S-1) + 0.0649(S-1)^2$	$1+\left(\frac{0.6155(S-n)+0.0649[(S-1)^2-(n-1)^2]}{1+0.6155(n-1)+0.0649(n-1)^2}\right)$
Logistic	$\mu$	1	-	$1+0.0050(S-1)$	$1+0.1666\left\{\frac{(S-n)}{0.3332+0.1666(n-1)}\right\}$
	0	$\sigma$	-	$1+0.1513(S-1)$	$1+0.2149\left\{\frac{(S-n)}{1.4189+0.2149(n-1)}\right\}$
	$\mu$	$\sigma$	-	$1+0.6516(S-1) + 0.0757(S-1)^2$	$1+\left(\frac{0.3081(S-n)+0.0358[(S-1)^2-(n-1)^2]}{0.4728+0.3081(n-1)+0.0358(n-1)^2}\right)$
Extreme-value	$\mu$	1	-	$1+0.4041(S-1)$	$1+0.4041\left\{\frac{(S-n)}{1+0.4041(n-1)}\right\}$
	0	$\sigma$	-	$1+0.2519(S-1)$	$1+0.2518\left\{\frac{(S-n)}{1+0.2518(n-1)}\right\}$
	$\mu$	$\sigma$	-	$1+0.6012(S-1) + 0.0686(S-1)^2$	$1+\left(\frac{0.6560(S-n)+0.1017[(S-1)^2-(n-1)^2]}{1+0.6560(n-1)+0.1017(n-1)^2}\right)$
Gamma	0	$\sigma$	2	$1+0.4393(S-1)$	$1+0.7296\left\{\frac{(S-n)}{1.6609+0.7296(n-1)}\right\}$
	0	$\sigma$	3	$1+0.4523(S-1)$	$1+1.1690\left\{\frac{(S-n)}{2.5846+1.1690(n-1)}\right\}$
	0	$\sigma$	4	$1+0.4591(S-1)$	$1+1.6161\left\{\frac{(S-n)}{3.5200+1.6161(n-1)}\right\}$
	0	$\sigma$	10	$1+0.4718(S-1)$	$1+4.2396\left\{\frac{(S-n)}{8.9820+4.2396(n-1)}\right\}$

**Example 2. (Linear Regression Model).** In this example, PROS( $n, S$ ) sampling design is applied to the simple regression model  $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$  with replicated observations of the response variable where for each value  $x_i$  of independent variable,  $i = 1, \dots, k$ , we have a PROS sample of  $Y$ 's



denoted by  $(Y_{i(d_1)}, \dots, Y_{i(d_n)})$ . For more details about the use of RSS sampling in this regression model, see Barreto and Barnett (1999) and Barabesi and El-Sharaawi (2001). Suppose  $\epsilon_i$  are independent and identically distributed random variables from a symmetric distribution with pdf  $f(\cdot)$  and cdf  $F(\cdot)$ , respectively. Let  $E(\epsilon_i) = 0$  and  $\text{Var}(\epsilon_i) = \sigma^2$ . Without loss of generality, we take  $\bar{x} = \frac{1}{k} \sum_{i=1}^k x_i = 0$ ,  $s_x^2 = \frac{1}{k} \sum_{i=1}^k x_i^2$  and let  $\boldsymbol{\theta} = (\beta_0, \beta_1, \sigma)$ . Using Example 1, it is easy to show that

$$\begin{aligned} \mathbb{I}_{srS}(\boldsymbol{\theta}) &= \sum_{i=1}^k \frac{n}{\sigma^2} \begin{pmatrix} E\{\frac{f'(Z)^2}{f(Z)^2}\} & x_i E\{\frac{f'(Z)^2}{f(Z)^2}\} & 0 \\ x_i E\{\frac{f'(Z)^2}{f(Z)^2}\} & x_i^2 E\{\frac{f'(Z)^2}{f(Z)^2}\} & 0 \\ 0 & 0 & E\{\frac{Z^2 f'(Z)^2}{f(Z)^2}\} - 1 \end{pmatrix} \\ &= \frac{nk}{\sigma^2} \text{diag} \left( E\{\frac{f'(Z)^2}{f(Z)^2}\}, s_x^2 E\{\frac{f'(Z)^2}{f(Z)^2}\}, E\{\frac{Z^2 f'(Z)^2}{f(Z)^2}\} - 1 \right), \end{aligned}$$

and

$$\begin{aligned} \mathbb{K}(\boldsymbol{\theta}) &= \sum_{i=1}^k \frac{2n(S-1)}{\sigma^2} \begin{pmatrix} E\{\frac{f(Z)^2}{F(Z)}\} & x_i E\{\frac{f(Z)^2}{F(Z)}\} & 0 \\ x_i E\{\frac{f(Z)^2}{F(Z)}\} & x_i^2 E\{\frac{f(Z)^2}{F(Z)}\} & 0 \\ 0 & 0 & E\{\frac{Z^2 f'(Z)^2}{F(Z)}\} \end{pmatrix} \\ &= \frac{2kn(S-1)}{\sigma^2} \text{diag} \left( E\{\frac{f(Z)^2}{F(Z)}\}, s_x^2 E\{\frac{f(Z)^2}{F(Z)}\}, E\{\frac{Z^2 f'(Z)^2}{F(Z)}\} \right). \end{aligned}$$

Note that  $RE_1(\boldsymbol{\theta})$  is independent of  $x_i$  and  $\boldsymbol{\theta}$  and it only depends on the pdf  $f(\cdot)$  and the corresponding cdf  $F(\cdot)$ . As a special case, when  $\epsilon_i$ s are normally distributed, one can easily show that

$$RE_1(\boldsymbol{\theta}) = \{1 + 0.4805(S-1)\}^2 \{1 + 0.1350(S-1)\}.$$

When  $S = n$ , we obtain the result of Barabesi and El-Sharaawi (2001) for RSS data as a special case of our results.

### 3.2 FI matrix of $\mathbf{X}_{pros}$ and the effect of misplacement errors

In this section we obtain the FI matrix of  $\mathbf{X}_{pros}$ . We study a setting when it is assumed that the subsetting process of PROS( $n, S$ ) design could be subjected to misplacement errors between the subset groups. For example, when the actual rank of a unit is in the judgment subset  $d_r$ , due to judgment ranking error it could be misplaced into another judgment subset, say  $d_s$ ,  $r \neq s$ , which leads to a different kind of ranking error than the one usually encountered in ranked set sampling. Note that the FI matrix of  $\mathbf{X}_{pros}$  under perfect subsetting assumption can also be obtained as a special case of the imperfect subsetting scenario. We use the missing data model proposed by Arslan and Ozturk (2013) to model possible misplacement errors in PROS sampling design. Let  $\mathbf{X}_{pros} = \{X_{[d_r]}, r = 1, \dots, n\}$  denote an imperfect PROS sample where  $[\cdot]$  is used to show the presence of misplacement errors in PROS subsetting process. When the subsetting process

is perfect we simply use  $X_{(d_r)}$  to show PROS observations. Let  $\alpha$  denote the misplacement probability matrix,

$$\alpha = \begin{bmatrix} \alpha_{d_1,d_1} & \alpha_{d_1,d_2} & \dots & \alpha_{d_1,d_n} \\ \alpha_{d_2,d_1} & \alpha_{d_2,d_2} & \dots & \alpha_{d_2,d_n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{d_n,d_1} & \alpha_{d_n,d_2} & \dots & \alpha_{d_n,d_n} \end{bmatrix}_{n \times n},$$

where  $\alpha_{d_r,d_h}$  is the misplacement probability of a unit from subset  $d_h$  into subset  $d_r$ . Since the design parameter  $D$  creates a partition over the sets, the matrix  $\alpha$  should be a double stochastic matrix such that  $\sum_{r=1}^n \alpha_{d_r,d_h} = \sum_{h=1}^n \alpha_{d_r,d_h} = 1$ . Suppose  $f_{[d_r]}(\cdot; \theta)$  is the pdf of  $X_{[d_r]}$ ,  $r = 1, \dots, n$ . One can easily show that

$$f_{[d_r]}(x_{[d_r]}; \theta) = \sum_{h=1}^n \alpha_{d_r,d_h} f_{(d_h)}(x_{[d_r]}; \theta) = f(x_{[d_r]}; \theta) g_r(x_{[d_r]}; \theta), \quad (8)$$

where

$$g_r(x; \theta) = n \sum_{h=1}^n \sum_{u \in d_h} \alpha_{d_r,d_h} \binom{S-1}{u-1} [F(x; \theta)]^{u-1} [1 - F(x; \theta)]^{S-u}. \quad (9)$$

The likelihood function under an imperfect PROS( $n, S$ ) design is now given by

$$L(\Omega) = \prod_{r=1}^n f_{[d_r]}(x_{[d_r]}; \theta) = \prod_{r=1}^n f(x_{[d_r]}; \theta) g_r(x_{[d_r]}; \theta),$$

where  $\Omega = (\theta, \alpha)$ . To obtain the FI matrix of an imperfect PROS sample and compare it with its SRS and RSS counterparts we need the following result, the proof of which is left to the reader.

**Lemma 2.** *Let  $Y_r = X_{[d_r]}$ ,  $r = 1, \dots, n$ , be observed from a continuous distribution with pdf  $f(\cdot; \theta)$  using an imperfect PROS( $n, S$ ) sampling design. Suppose  $f_{[d_r]}(\cdot; \theta)$  and  $g_r(\cdot, \theta)$  are defined as in (8) and (9), respectively. Under the regularity conditions of Chen et al. (2004), we have*

- (i)  $\sum_{r=1}^n f_{[d_r]}(x; \theta) = n f(x; \theta)$ ,
- (ii)  $\sum_{r=1}^n g_r(x; \theta) = n$ ,
- (iii)  $\sum_{r=1}^n E \left\{ \frac{D_{\theta}^2 g_r(Y_r; \theta)}{g_r(Y_r; \theta)} \right\} = 0$ ,
- (iv)  $\sum_{r=1}^n E \left\{ \frac{[D_{\theta} g_r(Y_r; \theta)][D_{\theta} g_r(Y_r; \theta)]^{\top}}{g_r^2(Y_r; \theta)} \right\} = \sum_{r=1}^n E \left\{ \frac{[D_{\theta} g_r(X; \theta)][D_{\theta} g_r(X; \theta)]^{\top}}{g_r(X; \theta)} \right\}$ .

Now, we show that the FI content of  $\mathbf{X}_{pros}$  is more than its SRS counterpart. Unfortunately, it is hard to obtain analytical results to compare the FI content of PROS and RSS data, therefore, we should rely on numerical studies for this case (see Tables 3 and 4).

**Theorem 3.** Under the conditions of Lemma 2, the FI matrix of an imperfect PROS( $n, S$ ) sample about unknown parameters  $\Omega = (\boldsymbol{\alpha}, \boldsymbol{\theta})$  is given by

$$\begin{aligned}\mathbb{I}_{ipros}(\Omega) &= \mathbb{I}_{srs}(\boldsymbol{\theta}) + \sum_{r=1}^n E \left\{ \frac{[D_{\boldsymbol{\theta}} g_r(X; \boldsymbol{\theta})][D_{\boldsymbol{\theta}} g_r(X; \boldsymbol{\theta})]^\top}{g_r(X; \boldsymbol{\theta})} \right\} \\ &= \mathbb{I}_{srs}(\boldsymbol{\theta}) + \sum_{r=1}^n \tilde{\Delta}_r,\end{aligned}$$

where  $\sum_{r=1}^n \tilde{\Delta}_r$  is a non-negative definite matrix.

*Proof.* The proof is similar to the proof of Theorem 1 and hence it is omitted.  $\square$

To study the effect of misplacement errors in the subsetting process of PROS( $n, S$ ) design on the information content of the sample, following Barabesi and El-Sharaawi (2001), we consider the following misplacement probabilities matrices when  $n = 2$  and  $n = 3$ ,

$$\boldsymbol{\alpha}_1 = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \quad \text{and} \quad \boldsymbol{\alpha}_2 = \begin{bmatrix} p & \frac{1-p}{2} & \frac{1-p}{2} \\ \frac{1-p}{2} & p & \frac{1-p}{2} \\ \frac{1-p}{2} & \frac{1-p}{2} & p \end{bmatrix}.$$

For some members of the location-scale family of distributions, numerical values of  $RE_1(\boldsymbol{\theta})$  and  $RE_2(\boldsymbol{\theta})$  are calculated to compare the FI content of imperfect PROS samples with their SRS and RSS counterparts of the same size when  $S = 6$  and  $S = 12$ . These values are reported in Tables 3 and 4, respectively. The results are calculated through a Monte Carlo simulation study comprising of 50,000 replications. Both tables show that misplacement errors in the subsetting process of PROS sampling have considerable effect on the information content of PROS data about the unknown parameters of the model. Note that, when the subsetting process is done randomly, i.e.,  $p = 1/2$  when  $n = 2$  and  $p = 1/3$  in the case  $n = 3$ , the FI content of PROS samples is the same as the FI content of SRS and RSS data of the same size. Similar results for comparing the FI content of imperfect RSS and SRS samples can be found in Barabesi and El-Sharaawi (2001).

Now, we investigate the effect of PROS sampling parameters  $S$  and  $n$  on the FI content of PROS samples compared with their RSS counterparts. To this end, we first calculate the FI content of two ranked set samples with fixed set sizes 6 and 12 when the cycle size is 1, under both perfect and different imperfect ranking scenarios. The FI content of RSS samples are then compared with that of PROS samples under different values of  $S, n$  and  $N$ , where  $N$  is the number of cycles in order to match the number of PROS observations with their corresponding RSS samples. Under some members of the location-scale family of distributions, Tables 7 and 8 provide the values of  $RE_2(\boldsymbol{\theta})$  for the sample sizes 6 and 12, respectively, where the subsetting and ranking error probability matrices are defined following the same structure used

in  $\alpha_1$  and  $\alpha_2$  with proper adjustments to the off-diagonal elements for the set size. For example, consider the case where  $S = 6, n = 3, l = 2$  in Table 7. In this case, RSS design with set size  $S = 6$  is compared with the PROS design with set size  $S = 6$ , each consisting of three subsets  $n = 3$  of equal sizes  $m = 2$ . Since the PROS design results in 3 observations (as opposed to RSS that results in 6 observations), PROS sampling is replicated with two cycles  $l = 2$ . The relative efficiency values are simulated through a Monte Carlo study with 50,000 replications. From Tables 7 and 8, it is at once apparent that sampling parameters  $S$  and  $n$  as well as ranking (subsetting) error models play key roles on the information content of PROS data about unknown parameters of the model. As noted earlier, one observes that the performance of  $\text{PROS}(n, S)$  and RSS coincides when  $S = n$ . We also note that for fixed set size  $S$  (in both RSS and PROS design) and under moderately accurate ranking in RSS design, some PROS samples carry less information than RSS of the same size about the parameter of the underlying population. However, the difference between the information content of PROS and RSS data diminishes as  $n$  increases to  $S$ . One may also observe more informative PROS samples than RSS data of the same size (even with a larger set size than that of PROS design) when the ranking error in RSS design is large.

### 3.3 FI using the Dell and Clutter model for misplacement ranking errors

Here, we propose two-stage Monte Carlo simulations to study the effect of misplacement ranking error models on the FI content of PROS samples. Following the model proposed in Dell and Clutter (1972), in the first stage we compute the misplacement probabilities of PROS and RSS designs. In the second stage, these misplacement probabilities are used to compute the FI content of PROS and RSS sampling designs. Using the Dell and Clutter model for  $\rho = 1, 0.9, 0.75, 0.5, 0.25$  (representing different degrees of association between the ranking covariate and the response variable), the first stage computes the misplacement probabilities matrices ( $\alpha_i = 1, \dots, 5$ ) for each  $\rho$  through simulations of size 5000. Using the estimated misplacement probabilities, in the second stage, we compute the FI content of the PROS, RSS and SRS sampling designs through Monte Carlo simulations comprising of 50,000 replicates. The results of the simulation studies for different family of distributions (like previous simulation studies) are reported in Tables 5 and 6. To explore the effect ranking errors on different distributions, we also computed the FI content of PROS samples under four different mixture of two univariate exponential distributions  $f(x; \Psi) = \pi\alpha e^{-\alpha x} + (1 - \pi)\beta e^{-\beta x}$ ,  $x > 0$ , where  $\pi \in (0, 1)$ ,  $\alpha, \beta > 0$  and  $\Psi = (\pi, \alpha, \beta)$ . To handle the mixture of exponential distributions, following Hill (1963), we calculated the numerical values of the relative efficiencies. To do so, a new parameter  $h = \frac{\alpha}{\beta}$  is introduced and the exponential mixture model with three parameters  $(\pi, \alpha, \beta)$  is transformed to a mixture density with two parameters  $(\pi, h)$ .

In the next section, we study the uncertainty structure (as another aspect of information content) of PROS samples in terms of some well-known measures including Shannon entropy, Rényi entropy and KL information.

Table 3: Values of  $RE_1$  and  $RE_2$  to compare the FI content of imperfect PROS data with its SRS and RSS counterparts of the same size for some distributions when  $S = 6$ .

Distribution	$n$	Design	$p$										
			0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Normal	2	$RE_1$	2.48	1.67	1.34	1.14	1.03	1.000	1.03	1.14	1.34	1.67	2.48
		$RE_2$	1.47	1.25	1.14	1.06	1.02	1.000	1.02	1.06	1.14	1.25	1.47
	3	$RE_1$	1.82	1.28	1.08	1.004	1.02	1.11	1.28	1.54	1.94	2.54	3.78
		$RE_2$	1.28	1.11	1.03	1.002	1.01	1.04	1.11	1.19	1.28	1.38	1.54
Exponential	2	$RE_1$	1.93	1.47	1.24	1.10	1.02	1.000	1.02	1.10	1.24	1.47	1.93
		$RE_2$	1.37	1.20	1.11	1.05	1.01	1.000	1.01	1.05	1.11	1.20	1.37
	3	$RE_1$	1.47	1.18	1.05	1.003	1.01	1.07	1.18	1.35	1.58	1.90	2.44
		$RE_2$	1.18	1.08	1.02	1.001	1.01	1.03	1.07	1.13	1.19	1.26	1.36
Logistic	2	$RE_1$	2.73	1.78	1.39	1.16	1.04	1.000	1.04	1.16	1.39	1.78	2.73
		$RE_2$	1.58	1.30	1.17	1.08	1.02	1.000	1.02	1.08	1.17	1.30	1.58
	3	$RE_1$	1.88	1.31	1.09	1.005	1.02	1.12	1.31	1.61	2.06	2.74	4.14
		$RE_2$	1.32	1.12	1.04	1.002	1.01	1.05	1.12	1.21	1.31	1.43	1.61

Table 4: Values of  $RE_1$  and  $RE_2$  to compare the FI content of imperfect PROS data with its SRS and RSS counterparts of the same size for some distributions when  $S = 12$ .

Distribution	$n$	Design	$p$										
			0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Normal	2	$RE_1$	3.15	1.96	1.48	1.20	1.05	1.000	1.05	1.20	1.48	1.96	3.15
		$RE_2$	1.87	1.46	1.26	1.12	1.03	1.000	1.03	1.12	1.26	1.46	1.87
	3	$RE_1$	2.51	1.49	1.13	1.007	1.03	1.18	1.46	1.90	2.56	3.58	5.74
		$RE_2$	1.77	1.29	1.09	1.005	1.02	1.11	1.26	1.46	1.68	1.93	2.32
Exponential	2	$RE_1$	2.39	1.69	1.35	1.15	1.04	1.000	1.04	1.15	1.35	1.69	2.39
		$RE_2$	1.70	1.38	1.21	1.10	1.02	1.000	1.02	1.10	1.21	1.38	1.70
	3	$RE_1$	1.85	1.31	1.09	1.005	1.02	1.12	1.30	1.57	1.93	2.43	3.30
		$RE_2$	1.48	1.19	1.06	1.003	1.01	1.08	1.18	1.31	1.45	1.61	1.82
Logistic	2	$RE_1$	3.56	2.14	1.57	1.24	1.06	1.000	1.06	1.24	1.57	2.14	3.56
		$RE_2$	2.06	1.56	1.32	1.15	1.04	1.000	1.04	1.15	1.32	1.56	2.06
	3	$RE_1$	2.72	1.55	1.15	1.008	1.03	1.20	1.53	2.04	2.81	4.04	6.65
		$RE_2$	1.90	1.33	1.10	1.005	1.02	1.13	1.30	1.53	1.79	2.09	2.56

Nevertheless, it is worth mentioning that the FI and uncertainty content play important roles in different inferential aspects of the PROS sampling designs including, for instance, maximum likelihood (ML) estimation and its properties. The FI matrix is a key concept in the theory of statistical inference particularly in the theory of ML estimation problem (Lehmann and Casella, 1998). It is used to derive asymptotic distribution of MLE and to calculate the

covariance matrices associated with ML estimates as well as Bayesian Statistics.

Table 5: Values of  $RE_1$  and  $RE_2$  to compare the FI content of imperfect PROS data with its SRS and RSS counterparts of the same size for some distributions based on different Dell-Clutter parameters when  $S \in \{6, 12\}$ .

Distribution	$n$	Design	S=6, $\rho$					S=12, $\rho$				
			0.25	0.50	0.75	0.90	1.00	0.25	0.50	0.75	0.90	1.00
Normal	2	$RE_1$	1.02	1.10	1.27	1.51	2.48	1.03	1.13	1.36	1.68	3.15
		$RE_2$	1.00	1.02	0.98	0.96	1.47	1.01	1.04	1.05	1.06	1.87
	3	$RE_1$	1.03	1.15	1.43	1.85	3.70	1.04	1.20	1.57	2.16	5.75
		$RE_2$	1.01	1.02	1.02	1.05	1.50	1.01	1.07	1.12	1.23	2.32
Exponential	2	$RE_1$	1.02	1.06	1.18	1.31	1.92	1.02	1.08	1.22	1.44	2.38
		$RE_2$	1.00	1.00	0.99	0.96	1.37	1.01	1.02	1.03	1.06	1.69
	3	$RE_1$	1.03	1.11	1.30	1.56	2.47	1.03	1.14	1.37	1.73	3.44
		$RE_2$	1.00	1.02	1.02	1.04	1.35	1.01	1.05	1.07	1.16	1.89
Logistic	2	$RE_1$	1.02	1.10	1.31	1.55	2.69	1.04	1.16	1.40	1.78	3.54
		$RE_2$	1.00	1.00	1.01	0.96	1.56	1.01	1.06	1.08	1.10	2.05
	3	$RE_1$	1.03	1.16	1.49	1.95	4.13	1.04	1.21	1.64	2.28	6.76
		$RE_2$	1.00	1.01	1.04	1.06	1.60	1.01	1.06	1.15	1.24	2.62
$\Psi = (\pi, h)$ (0.3, 1/3)	2	$RE_1$	1.04	1.13	1.44	1.80	3.83	1.05	1.21	1.51	2.02	5.11
		$RE_2$	1.01	0.98	0.95	0.90	2.36	1.02	1.04	1.00	1.00	3.16
	3	$RE_1$	1.06	1.22	1.59	2.07	4.51	1.06	1.23	1.65	2.30	6.44
		$RE_2$	1.01	1.02	1.00	1.00	1.91	1.02	1.03	1.04	1.11	2.73
$\Psi = (\pi, h)$ (0.3, 1/9)	2	$RE_1$	1.02	1.10	1.26	1.48	3.39	1.03	1.10	1.25	1.50	5.25
		$RE_2$	1.00	0.99	0.92	0.85	2.10	1.00	0.98	0.91	0.86	3.25
	3	$RE_1$	1.03	1.14	1.41	1.72	4.45	1.04	1.15	1.42	1.71	7.59
		$RE_2$	1.00	0.99	0.97	0.92	1.97	1.01	1.00	0.97	0.92	3.35
$\Psi = (\pi, h)$ (0.9, 1/3)	2	$RE_1$	1.05	1.19	1.50	2.02	3.67	1.05	1.22	1.61	2.16	4.34
		$RE_2$	1.01	1.02	0.97	0.91	2.15	1.01	1.05	1.04	0.98	2.54
	3	$RE_1$	1.04	1.20	1.56	2.13	5.24	1.06	1.25	1.70	2.39	6.60
		$RE_2$	1.00	1.01	0.97	1.01	2.05	1.02	1.05	1.06	1.13	2.59
$\Psi = (\pi, h)$ (0.9, 1/9)	2	$RE_1$	1.02	1.09	1.23	1.46	2.85	1.03	1.11	1.27	1.57	3.57
		$RE_2$	1.00	0.98	0.91	0.83	1.74	1.01	1.00	0.94	0.89	2.18
	3	$RE_1$	1.03	1.12	1.36	1.76	4.33	1.04	1.16	1.44	1.86	6.95
		$RE_2$	1.00	1.00	0.97	0.98	1.84	1.01	1.03	1.02	1.03	2.96

## 4 Other Information Criteria

The concept of information and uncertainty of random samples is so rich that several measures have been proposed to study different aspects of these concepts. For example, in the Engineering studies, the Shannon entropy, Rényi entropy and KL information measures are used more than FI to quantify the information and uncertainty structures of random samples. These measures quantify the amount of uncertainty inherent in the joint probability distribution of a random sample and have been applied in many areas such as ecological studies, computer sciences and information technology, in different contexts including order statistics, spacings, censored data, reliability, life testing, record data and text analysis. For more details see Jafari Jozani and Ahmadi (2014) and Johnson (2004) and references therein.

In this section, we compare the Shannon entropy, Rényi entropy and KL information of PROS data with SRS

Table 6: Values of  $RE_2$  to compare the FI content of imperfect PROS( $n, S$ ) with imperfect RSS of a fixed set size  $S \in \{6, 12\}$  under different Dell-Clutter Model

Distribution	$S$	$n$	$N$	$S = 6, \rho$					$S$	$n$	$N$	$S = 12, \rho$				
				0.25	0.50	0.75	0.90	1.00				0.25	0.50	0.75	0.90	1.00
Normal	4	2	3	0.97	0.88	0.73	0.58	0.39	6	2	6	0.97	0.84	0.61	0.40	0.16
	6	2	3	0.98	0.89	0.75	0.60	0.44	6	3	4	0.98	0.90	0.70	0.49	0.25
	6	3	2	0.99	0.94	0.86	0.75	0.67	12	2	6	0.97	0.87	0.67	0.45	0.21
	8	2	3	0.98	0.90	0.78	0.62	0.49	12	3	4	0.98	0.91	0.77	0.57	0.39
	12	2	3	0.99	0.91	0.82	0.68	0.56	12	4	3	0.99	0.95	0.83	0.68	0.55
	12	3	2	0.99	0.98	0.93	0.86	1.02	12	6	2	1.00	0.99	0.92	0.85	0.81
	12	6	1	1.01	1.03	1.11	1.26	2.04	12	12	1	1.00	1.01	1.01	1.03	1.03
Exponential	4	2	3	1.00	1.00	1.02	1.02	1.03	6	2	6	0.96	0.84	0.65	0.52	0.35
	6	2	3	0.98	0.90	0.78	0.69	0.65	6	3	4	0.97	0.87	0.73	0.62	0.45
	6	3	2	0.99	0.95	0.88	0.82	0.83	12	2	6	0.96	0.86	0.68	0.57	0.44
	8	2	3	0.98	0.91	0.80	0.71	0.70	12	3	4	0.97	0.91	0.76	0.67	0.62
	12	2	3	0.98	0.92	0.82	0.75	0.80	12	4	3	0.98	0.94	0.84	0.76	0.76
	12	3	2	0.99	0.97	0.92	0.90	1.14	12	6	2	0.99	0.97	0.91	0.87	0.88
	12	6	1	1.01	1.04	1.09	1.16	1.61	12	12	1	1.00	1.00	0.99	0.99	0.99
Logistic	4	2	3	0.97	0.88	0.71	0.56	0.36	6	2	6	0.96	0.82	0.59	0.38	0.16
	6	2	3	0.97	0.89	0.72	0.58	0.43	6	3	4	0.98	0.88	0.69	0.47	0.24
	6	3	2	0.99	0.94	0.85	0.74	0.66	12	2	6	0.96	0.87	0.64	0.43	0.20
	8	2	3	0.97	0.92	0.76	0.61	0.49	12	3	4	0.99	0.90	0.74	0.55	0.38
	12	2	3	0.98	0.93	0.80	0.65	0.56	12	4	3	0.99	0.92	0.81	0.66	0.54
	12	3	2	0.99	0.99	0.91	0.87	1.08	12	6	2	0.99	0.96	0.88	0.81	0.75
	12	6	1	1.00	1.03	1.13	1.26	2.12	12	12	1	1.00	1.00	1.00	0.98	0.99

Table 7: Values of  $RE_2$  to compare the FI content of imperfect PROS( $n, S$ ) with imperfect RSS of a fixed set size 6.

Distribution	$S$	$n$	$N$	$p$										
				0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Normal	4	2	3	1.75	1.49	1.25	1.03	0.83	0.67	0.56	0.48	0.42	0.38	0.37
	6	2	3	2.03	1.63	1.33	1.05	0.83	0.67	0.56	0.49	0.44	0.42	0.43
	6	3	2	1.49	1.24	1.07	0.93	0.82	0.74	0.69	0.66	0.64	0.63	0.65
	6	6	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	8	2	3	2.23	1.73	1.38	1.07	0.84	0.67	0.56	0.50	0.46	0.45	0.47
	12	2	3	2.59	1.92	1.48	1.11	0.85	0.67	0.57	0.52	0.49	0.49	0.54
	12	3	2	2.09	1.46	1.13	0.93	0.83	0.79	0.80	0.82	0.86	0.91	1.01
Exponential	12	6	1	1.22	1.03	1.01	1.08	1.20	1.36	1.51	1.66	1.80	1.93	2.05
	4	2	3	1.51	1.34	1.18	1.03	0.89	0.77	0.69	0.63	0.59	0.56	0.56
	6	2	3	1.71	1.44	1.23	1.05	0.89	0.77	0.69	0.64	0.61	0.61	0.64
	6	3	2	1.33	1.17	1.05	0.95	0.88	0.83	0.80	0.79	0.79	0.80	0.82
	6	6	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	8	2	3	1.89	1.54	1.28	1.07	0.90	0.77	0.70	0.65	0.64	0.65	0.70
	12	2	3	2.11	1.65	1.34	1.09	0.90	0.77	0.70	0.67	0.66	0.69	0.78
Logistic	12	3	2	1.66	1.30	1.09	0.96	0.89	0.87	0.88	0.91	0.96	1.02	1.11
	12	6	1	1.14	1.02	1.00	1.05	1.14	1.24	1.34	1.42	1.50	1.56	1.62
	4	2	3	1.89	1.57	1.30	1.04	0.83	0.66	0.55	0.47	0.42	0.38	0.37
	6	2	3	2.25	1.73	1.38	1.07	0.83	0.66	0.55	0.48	0.44	0.42	0.44
	6	3	2	1.55	1.27	1.08	0.93	0.82	0.74	0.69	0.67	0.65	0.65	0.67
	6	6	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	8	2	3	2.50	1.87	1.45	1.10	0.84	0.66	0.56	0.49	0.46	0.46	0.49
	12	2	3	2.94	2.08	1.56	1.14	0.85	0.66	0.56	0.51	0.50	0.51	0.58
	12	3	2	2.23	1.51	1.14	0.93	0.83	0.80	0.81	0.84	0.89	0.96	1.07
	12	6	1	1.23	1.03	1.01	1.09	1.22	1.39	1.57	1.73	1.89	2.03	2.17

Table 8: Values of  $RE_2$  to compare the FI content of imperfect PROS( $n, S$ ) with imperfect RSS of a fixed set size 12.

Distribution	$S$	$n$	$N$	$p$										
				0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Normal	6	2	6	2.24	1.67	1.17	0.78	0.52	0.36	0.27	0.21	0.18	0.16	0.16
	6	3	4	1.62	1.27	0.94	0.68	0.51	0.40	0.33	0.29	0.26	0.24	0.24
	12	2	6	2.86	1.98	1.31	0.82	0.52	0.36	0.27	0.22	0.20	0.19	0.21
	12	3	4	2.27	1.49	1.00	0.69	0.51	0.43	0.38	0.36	0.35	0.35	0.37
	12	4	3	1.78	1.23	0.89	0.69	0.59	0.53	0.50	0.49	0.49	0.50	0.52
	12	6	2	1.31	1.05	0.88	0.79	0.74	0.72	0.71	0.70	0.71	0.71	0.73
	12	12	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.99	0.99	0.99
Exponential	6	2	6	1.80	1.46	1.14	0.86	0.66	0.53	0.44	0.39	0.36	0.35	0.36
	6	3	4	1.38	1.18	0.97	0.79	0.65	0.56	0.51	0.47	0.46	0.45	0.46
	12	2	6	2.23	1.68	1.24	0.90	0.67	0.53	0.44	0.40	0.39	0.40	0.44
	12	3	4	1.78	1.33	1.01	0.79	0.66	0.59	0.56	0.56	0.57	0.59	0.64
	12	4	3	1.46	1.15	0.93	0.80	0.72	0.69	0.68	0.68	0.70	0.72	0.75
	12	6	2	1.19	1.03	0.93	0.87	0.84	0.83	0.84	0.85	0.86	0.87	0.89
	12	12	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Logistic	6	2	6	2.41	1.76	1.21	0.78	0.51	0.35	0.25	0.20	0.17	0.16	0.16
	6	3	4	1.69	1.30	0.95	0.67	0.50	0.39	0.32	0.28	0.26	0.24	0.24
	12	2	6	3.14	2.11	1.36	0.83	0.51	0.35	0.26	0.22	0.19	0.19	0.20
	12	3	4	2.43	1.55	1.00	0.68	0.50	0.42	0.37	0.36	0.35	0.36	0.38
	12	4	3	1.87	1.25	0.89	0.69	0.58	0.53	0.51	0.50	0.51	0.52	0.55
	12	6	2	1.33	1.05	0.88	0.79	0.74	0.72	0.71	0.72	0.72	0.73	0.75
	12	12	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

and RSS data of the same size. Throughout this section, the subsetting process of PROS design and the ranking process of RSS are assumed to be perfect.

#### 4.1 Shannon Entropy of the PROS sample

Let  $X$  be a continuous random variable with pdf  $f(\cdot, \theta)$ . The Shannon entropy associated with  $X$ , is defined as

$$H(X; \theta) = - \int f(x; \theta) \log f(x; \theta) dx,$$

subject to the existence of the integral. The Shannon entropy, as a quantitative measure of information (uncertainty), is extensively used in information technology, computer science and other engineering fields. In practice, smaller values of the Shannon entropy are more desirable (see Johnson, 2004). The Shannon entropy content of a SRS of size  $n$  is given by

$$H_n(\mathbf{X}_{srs}; \theta) = - \sum_{i=1}^n \int f(x; \theta) \log f(x; \theta) dx = n H(X_1; \theta).$$

Similarly, for an RSS of size  $n$  (with the set size  $n$ )

$$H_n(\mathbf{X}_{rss}; \theta) = - \sum_{i=1}^n \int f^{(i:n)}(x; \theta) \log f^{(i:n)}(x; \theta) dx,$$

where  $f^{(i:n)}(\cdot; \theta)$  is the pdf of the  $i$ -th order statistic in a SRS of size  $n$  from  $f(\cdot; \theta)$ . Furthermore, for a PROS( $n, S$ ) sample, it is easy to see that

$$H_n(\mathbf{X}_{pros}; \theta) = - \sum_{r=1}^n \int f_{(d_r)}(y; \theta) \log f_{(d_r)}(y; \theta) dy.$$



In the following lemma, we show that the Shannon entropy of PROS data is smaller than that of SRS data of the same size. Unfortunately, we were not able to obtain an ordering relationship among the Shannon entropy of RSS and PROS data of the same size. Instead, we obtain a lower bound for the Shannon entropy of a PROS( $n, S$ ) sample in terms of the Shannon entropy of an RSS data of size  $S$  when the set size is  $S$ .

**Lemma 3.** *Let  $\mathbf{X}_{pros}$  be a PROS( $n, S$ ) sample from a population with pdf  $f(\cdot; \boldsymbol{\theta})$  and let  $m = S/n$  be the number of observations in each subset. Suppose  $\mathbf{X}_{srs}$  is a SRS of size  $n$  from  $f(\cdot; \boldsymbol{\theta})$  with the Shannon entropy  $H_n(\mathbf{X}_{srs}; \boldsymbol{\theta})$  and  $H_S(\mathbf{X}_{rss}; \boldsymbol{\theta})$  represent the Shannon entropy of an RSS of size  $S$  when the set size is  $S$ . Then,*

$$\frac{1}{m}H_S(\mathbf{X}_{rss}; \boldsymbol{\theta}) \leq H_n(\mathbf{X}_{pros}; \boldsymbol{\theta}) \leq H_n(\mathbf{X}_{srs}; \boldsymbol{\theta}), \quad \text{for all } n \in N.$$

*Proof.* Using (3) and convexity of  $h(t) = t \log t, t > 0$ , we have

$$\begin{aligned} H_n(\mathbf{X}_{pros}; \boldsymbol{\theta}) &\leq -n \int \left( \frac{1}{n} \sum_{r=1}^n f_{(d_r)}(x; \boldsymbol{\theta}) \right) \left( \log \left[ \frac{1}{n} \sum_{r=1}^n f_{(d_r)}(x; \boldsymbol{\theta}) \right] \right) dx \\ &= H_n(\mathbf{X}_{srs}; \boldsymbol{\theta}). \end{aligned}$$

Furthermore, using (2) and convexity of  $h(t) = t \log t, t > 0$ , we have

$$\begin{aligned} H_n(\mathbf{X}_{pros}; \boldsymbol{\theta}) &= -\sum_{r=1}^n \int \left( \frac{1}{m} \sum_{u \in d_r} f^{(u:S)}(x; \boldsymbol{\theta}) \right) \left( \log \left[ \frac{1}{m} \sum_{u \in d_r} f^{(u:S)}(x; \boldsymbol{\theta}) \right] \right) dx \\ &\geq -\frac{1}{m} \sum_{r=1}^n \sum_{u \in d_r} \int f^{(u:S)}(x; \boldsymbol{\theta}) \log f^{(u:S)}(x; \boldsymbol{\theta}) dx \\ &= \frac{1}{m} H_S(\mathbf{X}_{rss}; \boldsymbol{\theta}), \end{aligned}$$

which completes the proof. □

## 4.2 Rényi entropy of PROS data

In this section we use the Rényi entropy as a quantitative measure of the entropy associated with PROS data  $\mathbf{X}_{pros}$ . The Rényi entropy of a random variable  $X$  with pdf  $f(\cdot; \boldsymbol{\theta})$  is defined as follows

$$H_\alpha(X; \boldsymbol{\theta}) = \frac{1}{1-\alpha} \log \mathbb{E}[f^{\alpha-1}(X; \boldsymbol{\theta})],$$

where  $\alpha > 0, \alpha \neq 1$ . The Rényi entropy is a very general measure and includes the Shannon entropy as its special case due to the following relationship

$$\lim_{\alpha \rightarrow 1} H_\alpha(X; \boldsymbol{\theta}) = - \int f(x; \boldsymbol{\theta}) \log f(x; \boldsymbol{\theta}) dx = H(X; \boldsymbol{\theta}).$$

Due to the flexibility of the Rényi entropy,  $H_\alpha(X; \boldsymbol{\theta})$  has been used in many fields such as statistics, ecology, engineering and etc. We derive the Rényi entropy of  $\mathbf{X}_{pros}$  and compare it with the Rényi entropy of  $\mathbf{X}_{srs}$ . We present the results for  $0 < \alpha < 1$  and the case with  $\alpha > 1$ , which requires further investigation, will be presented in later works. To this end, the Rényi entropy of a SRS of size  $n$  is given by

$$H_{\alpha,n}(\mathbf{X}_{srs}; \boldsymbol{\theta}) = \frac{1}{1-\alpha} \sum_{i=1}^n \log \int f^\alpha(x_i; \boldsymbol{\theta}) dx_i = n H_\alpha(X_1; \boldsymbol{\theta});$$

and for an RSS with set size  $n$ ,

$$H_{\alpha,n}(\mathbf{X}_{rss}; \boldsymbol{\theta}) = \frac{1}{1-\alpha} \sum_{i=1}^n \log \int [f^{(i:n)}(x; \boldsymbol{\theta})]^\alpha dx.$$

Also, for a PROS( $n, S$ ) sample, one gets

$$H_{\alpha,n}(\mathbf{X}_{pros}; \boldsymbol{\theta}) = \frac{1}{1-\alpha} \sum_{r=1}^n \log \int [f_{(d_r)}(x; \boldsymbol{\theta})]^\alpha dx.$$

**Lemma 4.** Let  $H_{\alpha,n}(\mathbf{X}_{pros}; \boldsymbol{\theta})$  represent the Rényi entropy of a PROS( $n, S$ ) sample of size  $n$  from a population with pdf  $f(\cdot; \boldsymbol{\theta})$ . Suppose  $\mathbf{X}_{srs}$  and  $\mathbf{X}_{rss}^*$  be a SRS of size  $n$  and an RSS of size  $S$  (with the set size  $S$ ) from  $f(\cdot; \boldsymbol{\theta})$ , respectively. For any  $0 < \alpha < 1$  and all  $n \in \mathbb{N}$ , we have

$$\frac{1}{m} H_{\alpha,S}(\mathbf{X}_{rss}^*; \boldsymbol{\theta}) \leq H_{\alpha,n}(\mathbf{X}_{pros}; \boldsymbol{\theta}) \leq H_{\alpha,n}(\mathbf{X}_{srs}; \boldsymbol{\theta}).$$

*Proof.* By using (2) and the concavity of the functions  $h_1(t) = \log t$  and  $h_2(t) = t^\alpha$ , we have

$$\begin{aligned} H_{\alpha,n}(\mathbf{X}_{pros}; \boldsymbol{\theta}) &\leq \frac{n}{1-\alpha} \left[ \log \int \frac{1}{n} \sum_{r=1}^n \left( \frac{1}{m} \sum_{u \in d_r} f^{(u:S)}(x; \boldsymbol{\theta}) \right)^\alpha dx \right] \\ &\leq \frac{n}{1-\alpha} \log \int \left( \frac{1}{S} \sum_{r=1}^n \sum_{u \in d_r} f^{(u:S)}(x; \boldsymbol{\theta}) \right)^\alpha dx \\ &= H_{\alpha,n}(\mathbf{X}_{srs}; \boldsymbol{\theta}). \end{aligned}$$

Similarly, one can show the following inequalities

$$\begin{aligned} H_{\alpha,n}(\mathbf{X}_{pros}; \boldsymbol{\theta}) &\geq \frac{1}{1-\alpha} \sum_{r=1}^n \log \left( \frac{1}{m} \sum_{u \in d_r} \int [f^{(u:S)}(x; \boldsymbol{\theta})]^\alpha dx \right) \\ &\geq \frac{1}{m(1-\alpha)} \sum_{r=1}^n \sum_{u \in d_r} \log \left( \int [f^{(u:S)}(x; \boldsymbol{\theta})]^\alpha dx \right) \\ &= \frac{1}{m} H_{\alpha,S}(\mathbf{X}_{rss}^*; \boldsymbol{\theta}), \end{aligned}$$

which complete the proof.  $\square$

### 4.3 KL Information of the PROS technique

The Kullback-Leibler (KL) discrepancy is another measure which can be used to quantify the information regarding a random phenomenon by comparing two probability density functions corresponding to a random experiment. Consider two pdfs  $f(\cdot; \boldsymbol{\theta})$  and  $g(\cdot; \boldsymbol{\theta})$ . The KL information measure based on  $f(\cdot; \boldsymbol{\theta})$  and  $g(\cdot; \boldsymbol{\theta})$  is defined by

$$K(f, g) = \int f(t; \boldsymbol{\theta}) \log \left( \frac{f(t; \boldsymbol{\theta})}{g(t; \boldsymbol{\theta})} \right) dt,$$

which quantifies the information lost by using  $g(\cdot; \boldsymbol{\theta})$  for the density of the random variable  $X$  instead of  $f(\cdot; \boldsymbol{\theta})$ . In this section, using the KL measure we make a comparison among PROS sampling, simple random sampling and ranked set sampling designs to determine which design provides more informative samples from the underlying population. To this end, we use

$$K(L_{pros}(\boldsymbol{\theta}|\mathbf{y}), L_{srs}(\boldsymbol{\theta}|\mathbf{y})) = \oint L_{pros}(\boldsymbol{\theta}|\mathbf{y}) \log \left( \frac{L_{pros}(\boldsymbol{\theta}|\mathbf{y})}{L_{srs}(\boldsymbol{\theta}|\mathbf{y})} \right) d\mathbf{y}, \quad (10)$$

to compare  $\text{PROS}(n, S)$  and simple random sampling designs, where  $L_{\text{pros}}(\boldsymbol{\theta}|\mathbf{y})$  and  $L_{\text{srs}}(\boldsymbol{\theta}|\mathbf{y})$  denote the likelihood functions of PROS and SRS data of the same size, respectively. The KL information measure for comparing ranked set sampling and simple random sampling is defined similarly by using (10) and setting  $S = n$  in PROS sampling design. One can interpret (10) in terms of a hypothesis testing problem within the Neyman-Pearson log-likelihood ratio testing framework (see Johnson, 2004).

**Lemma 5.** *Let  $L_{\text{pros}}(\boldsymbol{\theta}|\mathbf{y})$  and  $L_{\text{srs}}(\boldsymbol{\theta}|\mathbf{y})$  denote, respectively, the likelihood functions of a  $\text{PROS}(n, S)$  sample and a SRS of size  $n$  from a population with pdf  $f(\cdot; \boldsymbol{\theta})$ . Then we have*

$$K(L_{\text{pros}}(\boldsymbol{\theta}|\mathbf{y}), L_{\text{srs}}(\boldsymbol{\theta}|\mathbf{y})) = \sum_{r=1}^n \int f_{(d_r)}(y; \boldsymbol{\theta}) \log \left( \frac{f_{(d_r)}(y; \boldsymbol{\theta})}{f(y; \boldsymbol{\theta})} \right) dy.$$

*Proof.* To show the result, using (10) we have

$$\begin{aligned} K(L_{\text{pros}}(\boldsymbol{\theta}|\mathbf{y}), L_{\text{srs}}(\boldsymbol{\theta}|\mathbf{y})) &= \sum_{r=1}^n \oint \left\{ \prod_{h=1}^n f_{(d_h)}(y_h; \boldsymbol{\theta}) \right\} \log \left( \frac{f_{(d_r)}(y_r; \boldsymbol{\theta})}{f(y_r; \boldsymbol{\theta})} \right) \left\{ \prod_{j=1}^n dy_j \right\} \\ &= \sum_{r=1}^n \int f_{(d_r)}(y; \boldsymbol{\theta}) \log \left( \frac{f_{(d_r)}(y; \boldsymbol{\theta})}{f(y; \boldsymbol{\theta})} \right) dy; \end{aligned}$$

where the last equality follows from the independence of observations and the fact that  $n - 1$  of the integrals are 1.  $\square$

In the following lemma, we show that KL information distance between the likelihoods of PROS and SRS sampling designs is greater than the one between the likelihoods of two SRS sampling designs. Hence, PROS data are more informative than SRS data about the underlying population. We also obtain a lower bound for the KL information between the likelihoods of PROS and SRS data of the same size.

**Lemma 6.** *Let  $L_{\text{pros}}(\boldsymbol{\theta}|\mathbf{y})$  denote the likelihood function of a  $\text{PROS}(n, S)$  sample from a population with pdf  $f(\cdot, \boldsymbol{\theta})$ . Suppose  $L_{\text{srs},1}(\boldsymbol{\theta}|\mathbf{y})$  and  $L_{\text{srs},2}(\boldsymbol{\theta}|\mathbf{y})$  denote the likelihood functions of simple random samples of size  $n$  from  $f(\cdot; \boldsymbol{\theta})$  and  $g(\cdot; \boldsymbol{\theta})$ , respectively. In addition, let  $L_{\text{rss}^*}(\boldsymbol{\theta}|\mathbf{y})$  represent the likelihood function of a RSS of size  $S$  when the set size is  $S$ . Then,*

$$K(L_{\text{srs},1}(\boldsymbol{\theta}|\mathbf{y}), L_{\text{srs},2}(\boldsymbol{\theta}|\mathbf{y})) \leq K(L_{\text{pros}}(\boldsymbol{\theta}|\mathbf{y}), L_{\text{srs},2}(\boldsymbol{\theta}|\mathbf{y})) \leq \frac{1}{m} K(\tilde{L}_{\text{rss}^*}(\boldsymbol{\theta}|\mathbf{y}), L_{\text{srs},2}(\boldsymbol{\theta}|\mathbf{y})).$$

*Proof.* Applying Lemma 5 and using the convexity of  $h(t) = t \log t$ ,  $t > 0$ , we derive

$$\begin{aligned} K(L_{\text{pros}}(\boldsymbol{\theta}|\mathbf{y}), L_{\text{srs},2}(\boldsymbol{\theta}|\mathbf{y})) &= \sum_{r=1}^n \int g(y; \boldsymbol{\theta}) \left( \frac{f_{(d_r)}(y; \boldsymbol{\theta})}{g(y; \boldsymbol{\theta})} \right) \log \left( \frac{f_{(d_r)}(y; \boldsymbol{\theta})}{g(y; \boldsymbol{\theta})} \right) dy \\ &\geq n \int g(y; \boldsymbol{\theta}) \left[ \frac{1}{n} \sum_{r=1}^n \frac{f_{(d_r)}(y; \boldsymbol{\theta})}{g(y; \boldsymbol{\theta})} \right] \log \left[ \frac{\frac{1}{n} \sum_{r=1}^n f_{(d_r)}(y; \boldsymbol{\theta})}{g(y; \boldsymbol{\theta})} \right] dy \\ &= n \int f(y; \boldsymbol{\theta}) \log \left( \frac{f(y; \boldsymbol{\theta})}{g(y; \boldsymbol{\theta})} \right) dy \\ &= K(L_{\text{srs},1}(\boldsymbol{\theta}), L_{\text{srs},2}(\boldsymbol{\theta})), \end{aligned}$$

which shows the first inequality. Similarly,

$$\begin{aligned}
K(L_{pros}(\boldsymbol{\theta}|\mathbf{y}), L_{srs,2}(\boldsymbol{\theta}|\mathbf{y})) &= \sum_{r=1}^n \int g(y; \boldsymbol{\theta}) \left( \frac{1}{m} \sum_{u \in d_r} \frac{f^{(u:S)}(y; \boldsymbol{\theta})}{g(y; \boldsymbol{\theta})} \right) \log \left( \frac{1}{m} \sum_{u \in d_r} \frac{f^{(u:S)}(y; \boldsymbol{\theta})}{g(y; \boldsymbol{\theta})} \right) dy \\
&\leq \frac{1}{m} \sum_{v=1}^S \int f^{(v:S)}(y; \boldsymbol{\theta}) \log \left( \frac{f^{(v:S)}(x; \boldsymbol{\theta})}{g(y; \boldsymbol{\theta})} \right) dy \\
&= \frac{1}{m} K(L_{rss^*}(\boldsymbol{\theta}|\mathbf{y}), L_{srs,2}(\boldsymbol{\theta}|\mathbf{y})),
\end{aligned}$$

which completes the proof.  $\square$

## 5 Concluding Remarks

In this paper, we have considered the information content and uncertainty associated with PROS samples from a population. First, we have compared the FI content of PROS samples with the FI content of SRS and RSS data of the same size under both perfect and imperfect subsetting assumptions. We showed that PROS sampling design results in more informative observations from the underlying population than simple random sampling and ranked set sampling. Some examples are presented to show the amount of the extra information provided by PROS sampling design. We have then considered other information and uncertainty measures such as the Shannon entropy, Rényi entropy and the KL information measures. Similar results have been obtained under the perfect subsetting assumption. It would naturally be of interest to extend these results to imperfect subsetting situations. The results of this paper suggest that one might be able to obtain more powerful tests for testing hypothesis or model selection problems based on PROS data. For example, it seems promising to develop goodness of fit tests based on PROS data under KL information measure. We believe that further investigation of PROS sampling design under the missing information criterion as in Hatefi and Jafari Jozani (2013) is of interest and appealing as well.

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## Appendix:

### FI of unbalanced PROS and the effect of misplacement errors

In this section, we study the FI matrix of the unbalanced PROS sampling design in a general setting when the subsets are allowed to be of different sizes. To obtain an unbalanced PROS sample, we first need to determine the sample of size  $K$  and set size  $S$ . Judgment sub-setting process is then applied to create  $K$  sets. We group these  $K$  sets into  $N$  cycles  $G_i = \{S_{1,i}, \dots, S_{n_i,i}\}; i = 1, \dots, N$ , where  $\sum_{i=1}^N n_i = K$ . Let  $D_{r,i} = \{d_{r[1]i}, \dots, d_{r[n_i]i}\}$  be the design parameter associated with set  $S_{r,i}$ , where  $d_{r[l]i}; l = 1, \dots, n_i$  is the  $l$ -th judgment subset in the set  $S_{r,i}$ . In

each cycle  $G_i; i = 1, \dots, N$ , we randomly select a unit from one of the sets (particularly from the judgment subset  $d_{r[r]i}; r = 1, \dots, n_i$ ) for full measurement, say  $X_{[d_r]i}$  and the number of unranked units in subset  $d_{r[r]i}$  is denoted by  $m_{ri}; r = 1, \dots, n_i; i = 1, \dots, N$ . To this end, the collection of measured observations  $\{X_{[d_r]i}; r = 1, \dots, n_i; i = 1, \dots, N\}$  is an unbalanced PROS sample of size  $K = \sum_{i=1}^N n_i$ . Table 9 illustrates the construction of an unbalanced PROS sample of size of  $K = 5$  with set size  $S = 6$  and cycle size  $N = 2$  so that in the first cycle we declare three subsets  $n_1 = 3$  and two subsets  $n_2 = 2$  of different sizes in the first and second cycles, respectively. In each set,  $m_{ri}$  represent the number of unranked units in the selected subset. For more details see Ozturk (2011).

Table 9: An example of unbalanced PROS design when  $S = 6, K = 5, N = 2, n_1 = 3, n_2 = 2$  and  $m_{ri}$  represents size of the selected subset in each set.

cycle	set	Subsets	$m_{ri}$	Observation
1	$S_{1,1}$	$D_{1,1} = \{\mathbf{d}_{1[1]1}, d_{1[2]1}, d_{1[3]1}\} = \{\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}, \{4, 5\}, \{6\}\}$	3	$X_{[d_1]1}$
	$S_{2,1}$	$D_{2,1} = \{d_{2[1]1}, \mathbf{d}_{2[2]1}, d_{2[3]1}\} = \{\{1, 2, 3\}, \{\mathbf{4}, \mathbf{5}\}, \{6\}\}$	2	$X_{[d_2]1}$
	$S_{3,1}$	$D_{3,1} = \{d_{3[1]1}, d_{3[2]1}, \mathbf{d}_{3[3]1}\} = \{\{1, 2, 3\}, \{4, 5\}, \{\mathbf{6}\}\}$	1	$X_{[d_3]1}$
2	$S_{1,2}$	$D_{1,2} = \{\mathbf{d}_{1[1]2}, d_{1[2]2}\} = \{\{\mathbf{1}, \mathbf{2}\}, \{3, 4, 5, 6\}\}$	2	$X_{[d_1]2}$
	$S_{2,2}$	$D_{2,2} = \{d_{2[1]2}, \mathbf{d}_{2[2]2}\} = \{\{1, 2\}, \{\mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}\}$	4	$X_{[d_2]2}$

We first present the following result.

**Lemma 7.** Let  $Y_{ri} = X_{[d_r]i}$  be an observation from unbalanced PROS sampling design from a continuous distribution with pdf  $f(\cdot; \theta)$ . With the knowledge of the design parameter  $D_{r,i}$ , the pdf of  $Y_{ri}$  is given by

$$f_{[r;m_{ri}]}(y; \theta) = \frac{1}{m_{ri}} \sum_{v \in d_{r[r]i}} f^{[v:S]}(y; \theta),$$

where  $f^{[v:S]}(y; \theta)$  is the pdf of the  $v$ -th judgment order statistics between  $S$  data.

*Proof.* For each  $Y_{ri}$  define the latent vector  $\Delta^{[d_r]i} = (\Delta^{[d_r]i}(v), v \in d_{r[r]i})$ , where

$$\Delta^{[d_r]i}(v) = \begin{cases} 1 & \text{if } Y_{ri} \text{ is selected from the } v\text{-th position within the subset } d_{r[r]i}; \\ 0 & \text{otherwise,} \end{cases}$$

with  $\sum_{v \in d_{r[r]i}} \Delta^{[d_r]i}(v) = 1$ . The joint pdf of  $(Y_{ri}, \Delta^{[d_r]i})$  is given by

$$f(y, \delta^{[d_r]i}; \theta) = \prod_{r=1}^{n_i} \prod_{v \in d_{r[r]i}} \left\{ \frac{1}{m_{ri}} f^{[v:S]}(y; \theta) \right\}^{\delta^{[d_r]i}(v)}.$$

Furthermore, by summing the joint distribution of  $(Y_{ri}, \Delta^{[d_r]i})$  over  $\Delta^{[d_r]i} = \delta^{[d_r]i}$ , the marginal distribution of  $Y_{ri}$  is obtained as follows

$$f_{[r;m_{ri}]}(y; \theta) = \sum_{\delta^{[d_r]i}} f(y, \delta^{[d_r]i}; \theta) = \frac{1}{m_{ri}} \sum_{v \in d_{r[r]i}} f^{[v:S]}(y; \theta).$$

□

Using Lemma 7, the likelihood function under an unbalanced PROS design is now given by

$$\begin{aligned} L(\Omega) &= \prod_{i=1}^N \prod_{r=1}^{n_i} f_{[r;m_{ri}]}(y_{ri}; \boldsymbol{\theta}) = \prod_{i=1}^N \prod_{r=1}^{n_i} \left\{ \frac{1}{m_{ri}} \sum_{v \in d_{r[r]i}} f^{[v:S]}(y_{ri}; \boldsymbol{\theta}) \right\} \\ &= \prod_{i=1}^N \prod_{r=1}^{n_i} \left\{ \frac{1}{m_{ri}} \sum_{v \in d_{r[r]i}} \sum_{h=1}^{n_i} \sum_{u \in d_{h[h]i}} \frac{\alpha_{[d_r, d_h]i}}{m_{hi}} f^{(u:S)}(y_{ri}; \boldsymbol{\theta}) \right\}, \end{aligned} \quad (11)$$

where  $\Omega = (\boldsymbol{\theta}, \boldsymbol{\alpha})$ ,  $f^{(u:S)}(\cdot; \boldsymbol{\theta})$  is the pdf of the  $u$ -th order statistics and in a similar vein to Subsection 3.2,  $\alpha_{[d_r, d_h]i}$  is considered as the misplacement probability of a unit from subset  $d_{h[h]i}$  into subset  $d_{r[r]i}$  so that  $\sum_{h=1}^{n_i} \alpha_{[d_r, d_h]i} = \sum_{r=1}^{n_i} \alpha_{[d_r, d_h]i} = 1; i = 1, \dots, N$ . Similarly, one can re-write the likelihood function (11) as follows

$$L(\Omega) = \prod_{i=1}^N \prod_{r=1}^{n_i} f_{[r;m_{ri}]}(y_{ri}; \boldsymbol{\theta}) = \prod_{i=1}^N \prod_{r=1}^{n_i} f(y_{ri}; \boldsymbol{\theta}) g_{ri}(y_{ri}; \boldsymbol{\theta}),$$

where

$$g_{ri}(y; \boldsymbol{\theta}) = \sum_{h=1}^{n_i} \sum_{u \in d_{h[h]i}} \alpha_{[d_r, d_h]i} \frac{S}{m_{hi}} \binom{S-1}{u-1} [F(y; \boldsymbol{\theta})]^{u-1} [1 - F(y; \boldsymbol{\theta})]^{S-u}. \quad (12)$$

Similar to Subsection 3.2, to obtain the FI matrix of an unbalanced PROS sample and compare it with its SRS and RSS counterparts one can easily obtain the following result.

**Lemma 8.** *Let  $Y_{r,i} = X_{[d_r]i}$ ,  $r = 1, \dots, n_i; i = 1, \dots, N$ , be observed from a continuous distribution with pdf  $f(\cdot; \boldsymbol{\theta})$  using an unbalanced PROS sampling design. Suppose  $f_{[r;m_{ri}]}(\cdot; \boldsymbol{\theta})$  and  $g_{ri}(\cdot; \boldsymbol{\theta})$  are defined as in Lemma 7 and (12), respectively. Under the regularity conditions of Chen et al. (2004), we have*

$$\begin{aligned} (i) \quad & \sum_{i=1}^N \sum_{r=1}^{n_i} E \left\{ \frac{D_{\boldsymbol{\theta}}^2 g_{ri}(X_{[d_r]i}; \boldsymbol{\theta})}{g_{ri}(X_{[d_r]i}; \boldsymbol{\theta})} \right\} = \sum_{i=1}^N \sum_{r=1}^{n_i} E \left\{ D_{\boldsymbol{\theta}}^2 g_{ri}(X; \boldsymbol{\theta}) \right\}, \\ (ii) \quad & \sum_{i=1}^N \sum_{r=1}^{n_i} E \left\{ \frac{[D_{\boldsymbol{\theta}} g_{ri}(X_{[d_r]i}; \boldsymbol{\theta})][D_{\boldsymbol{\theta}} g_{ri}(X_{[d_r]i}; \boldsymbol{\theta})]^\top}{g_{ri}^2(X_{[d_r]i}; \boldsymbol{\theta})} \right\} = \sum_{i=1}^N \sum_{r=1}^{n_i} E \left\{ \frac{[D_{\boldsymbol{\theta}} g_{ri}(X; \boldsymbol{\theta})][D_{\boldsymbol{\theta}} g_{ri}(X; \boldsymbol{\theta})]^\top}{g_{ri}(X; \boldsymbol{\theta})} \right\}. \end{aligned}$$

Now, we can present the main result of this section as follows.

**Theorem 4.** *Under the conditions of Lemma 8, the FI matrix of an unbalanced PROS sample about unknown parameters  $\Omega = (\boldsymbol{\alpha}, \boldsymbol{\theta})$  is given by*

$$\mathbb{I}_{upros}(\Omega) = \mathbb{I}_{srs}(\boldsymbol{\theta}) - \sum_{i=1}^N \sum_{r=1}^{n_i} E \left\{ D_{\boldsymbol{\theta}}^2 g_{ri}(X; \boldsymbol{\theta}) \right\} + \sum_{i=1}^N \sum_{r=1}^{n_i} E \left\{ \frac{[D_{\boldsymbol{\theta}} g_{ri}(X; \boldsymbol{\theta})][D_{\boldsymbol{\theta}} g_{ri}(X; \boldsymbol{\theta})]^\top}{g_{ri}(X; \boldsymbol{\theta})} \right\}.$$

Table 10 shows the FI content of unbalanced PROS samples compared with their SRS and RSS counterparts in the case of normal distribution and when  $N = 1$ ,  $S = 6$  and three subsets  $n = 3$  of different sizes have been declared. The misplacement ranking error models are obtained following the model proposed in Dell and Clutter (1972) when  $\rho \in \{0.25, 0.5, 0.75, 0.9, 1\}$ .

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Table 10: Values of  $RE_1$  and  $RE_2$  to compare the FI content of unbalanced PROS data with its SRS and RSS counterparts of the same size for normal distribution when  $S = 6$  and  $n \in \{2, 3\}$ .

$D = \{d_1, \dots, d_n\}$	Design	$\rho$				
		0.25	0.50	0.75	0.90	1.00
$\{\{1, 2, 3, 4, 5\}, \{6\}\}$	$RE_1$	1.134	1.823	3.094	4.754	8.026
	$RE_2$	1.110	1.666	2.412	3.006	4.768
$\{\{1, 2, 3, 4\}, \{5, 6\}\}$	$RE_1$	1.038	1.151	1.343	1.510	1.613
	$RE_2$	1.018	1.064	1.046	0.962	0.968
$\{\{1, 2, 3\}, \{4, 5, 6\}\}$	$RE_1$	1.020	1.095	1.271	1.513	2.507
	$RE_2$	1.002	1.013	0.993	0.959	1.494
$\{\{1, 2\}, \{3, 4, 5, 6\}\}$	$RE_1$	1.040	1.198	1.361	1.547	1.597
	$RE_2$	1.020	1.094	1.058	0.980	0.945
$\{\{1\}, \{2, 3, 4, 5, 6\}\}$	$RE_1$	1.137	1.796	3.170	4.748	8.175
	$RE_2$	1.120	1.654	2.467	3.021	4.859
$\{\{1\}, \{2\}, \{3, 4, 5, 6\}\}$	$RE_1$	1.071	1.485	2.196	2.927	3.389
	$RE_2$	1.052	1.331	1.599	1.688	1.374
$\{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$	$RE_1$	1.169	1.444	2.259	3.261	5.810
	$RE_2$	1.139	1.263	1.550	1.829	2.301
$\{\{1\}, \{2, 3, 4\}, \{5, 6\}\}$	$RE_1$	1.120	1.385	2.513	3.620	5.900
	$RE_2$	1.079	1.228	1.738	2.063	2.411
$\{\{1\}, \{2, 3, 4, 5\}, \{6\}\}$	$RE_1$	1.204	2.039	4.263	7.090	16.439
	$RE_2$	1.186	1.787	3.018	3.962	6.604
$\{\{1, 2\}, \{3, 4, 5\}, \{6\}\}$	$RE_1$	1.038	1.544	2.484	3.604	5.734
	$RE_2$	1.004	1.373	1.761	2.023	2.278
$\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$	$RE_1$	1.032	1.158	1.453	1.865	3.785
	$RE_2$	1.005	1.025	1.036	1.045	1.513
$\{\{1, 2, 3\}, \{4\}, \{5, 6\}\}$	$RE_1$	0.979	0.923	0.918	1.129	2.809
	$RE_2$	0.932	0.813	0.652	0.642	1.127
$\{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$	$RE_1$	0.994	0.939	0.946	1.089	2.874
	$RE_2$	0.961	0.845	0.681	0.606	1.143
$\{\{1, 2, 3, 4\}, \{5\}, \{6\}\}$	$RE_1$	1.086	1.378	2.178	2.955	3.463
	$RE_2$	1.077	1.203	1.553	1.685	1.386

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